Higher Cats RG talk 13(?) 11/19 notes. (§4.2–4.3)

[HTT]
(Rezk – Intro to Quasicats]
[Cisinski].

§4.2: Mapping spaces as fibers.

Recall the join of ssets X, Y ...

$$(X \star Y)_n = \coprod_{i+j+1<n} X_i \times Y_j$$

i.e. $\Delta^n \xrightarrow{f} X \star Y$ is
- an i-simplex in X
- a j-simplex in Y

This forms a functor $\forall X \in sSet$, $-\star X : sSet \longrightarrow sSet_{X/}$
$X \star - : sSet \longrightarrow sSet_{X/}$

These have R-adjoints ...

$$\begin{tikzcd} sSet \arrow[r, "-\star X", bend left] \arrow[r, phantom, "\perp"] & sSet_{X/} \arrow[l, "\text{slice}", bend left] \end{tikzcd} \qquad \begin{tikzcd} sSet \arrow[r, "X\star -", bend left] & sSet_{X/} \arrow[l, "\text{slice}", bend left] \end{tikzcd}$$

$S_{/p} \longleftarrow\!\shortmid \left(X \xrightarrow{p} S\right)$ $\qquad$ $S_{p/} \longleftarrow\!\shortmid \left(X \xrightarrow{p} S\right)$

$$\operatorname{Hom}_{sSet_{X/}}(T \star X, S) \cong \operatorname{Hom}_{sSet}(T, S_{/p}) \qquad (X \xrightarrow{p} S)$$

$$\begin{tikzcd} \Delta^0 \star X \arrow[r] & S \\ X \arrow[u, hook] \arrow[ur, "p"'] \end{tikzcd} \qquad \Delta^0 \to S_{/p}$$

$(S_{/p})_n = \operatorname{Hom}_{sSet_{X/}}(\Delta^n \star X, S)$ $\quad \left(X \xrightarrow{p} S\right)$

$= \operatorname{Hom}_{sSet}(\Delta^n, S_{/p})$

i.e. map $\Delta^n \star X \to S$ s.t.

$$\begin{tikzcd} \Delta^n \star X \arrow[r] & S \\ X \arrow[u, hook] \arrow[ur, "p"'] \arrow[u, phantom, "=", shift left=4] \end{tikzcd}$$

Note any $S_{/p}$ comes w/ a projection $S_{/p} \longrightarrow S$

$$\begin{tikzcd} \Delta^n \arrow[r, hook] \arrow[dr, "\varphi|_{\Delta^n}"'] & \Delta^n \star X \arrow[d, "\varphi"] \\ & S \end{tikzcd} \mapsto \varphi|_{\Delta^n}.$$

This lets us describe "mapping spaces" in an ∞-cat...

Def. Let $\mathcal{C}$: q cat.
$x, y \in \mathcal{C}_0$.
Then there's a sset $\operatorname{Map}^L_{\mathcal{C}}(x,y)$

$$\begin{tikzcd} \operatorname{Map}^L_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_{/y} \arrow[d] \\ \Delta^0 \arrow[r, "x"] & \mathcal{C} \end{tikzcd}$$

Similarly.–

$$\begin{tikzcd} \operatorname{Map}^R_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"] & \mathcal{C} \end{tikzcd}$$

Prop [HTT, 1.2.2.3]: $\operatorname{Map}^L_{\mathcal{C}}(x,y)$ & $\operatorname{Map}^R_{\mathcal{C}}(x,y)$ are Kan complexes.

Rk: Can show these are weakly equiv. to eachother & to the mapping space we saw before

$$\begin{tikzcd} \operatorname{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\Delta^1} \arrow[d] \\ \Delta^0 \arrow[r, "(x,y)"] & \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

[HTT, 4.2.1.8]

[C, 4.2.10].

$$\operatorname{Map}^L_{\mathcal{C}}(x,y) \xrightarrow{\sim} \operatorname{Map}_{\mathcal{C}}(x,y) \xleftarrow{\sim} \operatorname{Map}^R_{\mathcal{C}}(x,y).$$

To do this,
Cisinski defines an "alternate slice" (see Rezk, §10)
which is in some sense equivalent to the usual slice,
but useful in showing the equivalences.

Def (alt. join): Let $X, Y \in sSet$. The alt. join $X \diamond Y$ is the p.o.

$$\begin{tikzcd} X\times\partial\Delta^1\times Y \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & X\amalg Y \arrow[d] \\ X\times\Delta^1\times Y \arrow[r] & X\diamond Y \end{tikzcd}$$

$$X\times\partial\Delta^1\times Y \cong (X\times Y)\amalg(X\times Y) \xrightarrow{pr_1 \amalg pr_2} X\amalg Y$$

Similarly to the ordinary join there are alt slices which we can define as R-adjoints.

$$\begin{tikzcd} sSet \arrow[r, shift left, "-\diamond X"] & sSet_{X/} \arrow[l, shift left, "\text{alt slice}"] \end{tikzcd} \qquad \begin{tikzcd} sSet \arrow[r, shift left, "X\diamond -"] & sSet_{X/} \arrow[l, shift left, "\text{alt slice}"] \end{tikzcd}$$

$S^{/p}$ or $S_{//p}$ $\longleftarrow\!\shortmid$ $\left(X \xrightarrow{p} S\right)$ $\qquad$ $S^{p/}$ or $S_{p//}$ $\longleftarrow\!\shortmid$ $\left(X \xrightarrow{p} S\right)$

Prop. There's a weak cat'l equiv. $X\diamond Y \xrightarrow{\gamma_{X,Y}} X\star Y$.

w.e. in $sSet_{Joyal}$

Prop: For $X \in qCat$, $t: T \to X$ a map of ssets, there's an equiv. of ∞-cats

$$X_{/t} \xrightarrow{\sim} X^{/t}.$$

↳ lets you define Map^L as a fiber of $X_{/y}$ & Map as a fiber of $X^{/y}$ & lets you compare them.

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§4.3: Final objects.

Def. Let $X \in sSet$. An object $x \in X_0$ is a final object if the map $\Delta^0 \xrightarrow{x} X$ is final in the sense of [4.1.8]: $\forall$ sSet S, & morphism $X \xrightarrow{\varphi} S$, the map

$$\left(\begin{smallmatrix}\Delta^0 \\ \downarrow \varphi\circ x \\ S\end{smallmatrix}\right) \longrightarrow \left(\begin{smallmatrix}X \\ \downarrow \varphi \\ S\end{smallmatrix}\right)$$

is a weak equiv. in $sSet_{/S}$, contravariant

i.e. the induced map

$$\Delta^{0,\triangleright} \cup_{\Delta^0} S \longrightarrow X^{\triangleright} \cup_X S$$

is a weak categorical equivalence (see [HTT, 2.1.4])

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & S \arrow[d] \arrow[ddr, bend left] & & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S \arrow[d] \\ \Delta^{0,\triangleright} \arrow[r] \arrow[d, "x\star id"] & (p.o.) \arrow[dr] & & X^{\triangleright} \arrow[r] & (p.o.) \\ X^{\triangleright} \arrow[rr, bend right] & & (p.o.) & & \end{tikzcd}$$

Rk. $x \in X_0$ is final iff the map $\Delta^0 \xrightarrow{x} X$ is a R-anodyne ext.

Prop: Let $X \xrightarrow{f} Y$ be a map of ssets.
$x \in X_0$ a final object.

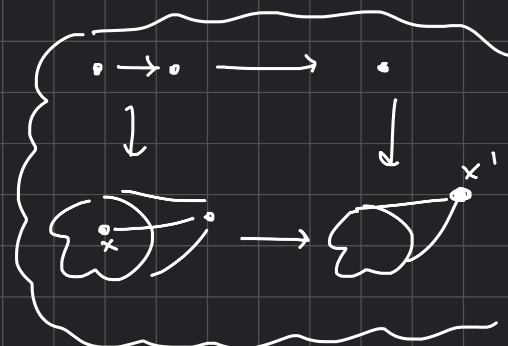
Then f is final in $sSet_{/Y,\,contra}$ iff $f(x)$ is a final object in Y.

Pf: $\Rightarrow$: Say f is final in $sSet_{/Y,\,contra}$. WTS: $\Delta^0 \xrightarrow{f(x)} Y$ is final.

$$\begin{tikzcd} \Delta^0 \arrow[rr, "f(x)"] \arrow[dr, "x"'] & & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

Both x & f are final $\Rightarrow$ by 4.1.9 (a).

$\Leftarrow$: Say $f(x)$ is final. Then by 4.1.9(b) . . . $\square$



Def. Let (X, x) be a ptd sSet.
Define a ptd sset $C(X)$ as a pushout
$= (X \star \Delta^0, x')$

$$\begin{tikzcd} \Delta^0 \star \Delta^0 \arrow[r] \arrow[d, "x \star id"'] & \Delta^0 \arrow[d, "x'"] \\ X \star \Delta^0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & C(X) \end{tikzcd}$$

$X \star \Delta^0 = X^{\triangleright}$

This forms a functor $C: sSet_* \longrightarrow sSet_*$.
$(X, x) \mapsto (CX, x')$

Prop: The object $x' : \Delta^0 \longrightarrow C(X)$ is final in $C(X)$.

Pf: $\Delta^0 \xrightarrow{x'} C(X)$ is a mono. since it's a p.o. of a mono.

So (x' is final) iff (it's a R-anodyne ext.) [4.1.9]

iff ($\Delta^0 \star \Delta^0 \to X \star \Delta^0$ is a R-anodyne ext.)

iff ($\Delta^0 \xrightarrow{x} X$ is a R-anodyne ext.)

iff (for any

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "x'"'] & A \arrow[d] \\ C(X) \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

$$\begin{tikzcd} \Delta^0 \star \Delta^0 \arrow[r] \arrow[d] & A \arrow[d, "p"] \\ X \star \Delta^0 \arrow[r] & B \end{tikzcd} \quad (\sim) \quad \begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "x"'] & A_{/a} \arrow[d] \\ X \arrow[r] & B_{/p(a)} \end{tikzcd}$$

The functor C has a R-adjt $sSet_* \longrightarrow sSet_*$

$$(Y,y) \longmapsto (Y_{/y}, id_y)$$

i.e. $\mathrm{Hom}_{sSet_*}(C(X), Y) \cong \mathrm{Hom}_{sSet_*}(X, Y_{/y})$

Prop. Let $X \in sSet$, $x \in X_0$.

If there's a section s

$$\begin{tikzcd} X_{/x} \arrow[d] \\ X \arrow[u, bend right, "s"'] \end{tikzcd}$$

s.t. $s(x) = id_x$

$\Rightarrow$ x is a final object.

If $X \in qCat$, then this is an iff.

Pf. $\Rightarrow$: If a section exists, then by adjunction

$$\mathrm{Hom}_{sSet_*}(X, X_{/x}) \cong \mathrm{Hom}_{sSet_*}(CX, X)$$

$$s \longleftrightarrow r : CX \to X$$

s.t. $r(x') = x$

$$\begin{tikzcd} \Delta^0 \arrow[rr, "x"] \arrow[dr, "x'"'] & & X \\ & C(X) \arrow[ur, "r"'] & \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} \Delta^0 \arrow[r] \arrow[d, "x"'] & \Delta^0 \arrow[r] \arrow[d, "x'"] & \Delta^0 \arrow[d, "x"] \\ X \arrow[r] & CX \arrow[r, "r"'] & X \end{tikzcd}$$

$\begin{matrix}\Delta^0 \\ \downarrow \\ X\end{matrix}$ is a retract of $\begin{matrix}\Delta^0 \\ \downarrow x' \\ CX\end{matrix}$

$\Rightarrow$ x is final. //

If $X \in qCat$ & $x \in X_0$ is final... WTS $\exists$ a section $X \to X_{/x}$

$$\Leftarrow: \begin{tikzcd} \Delta^0 \arrow[r, "id_x"] \arrow[d, "x"'] & X_{/x} \arrow[d] \\ X \arrow[r, "id_X"'] \arrow[ur, "s"] & X \end{tikzcd}$$

$\begin{matrix}X_{/x} \\ \downarrow \\ X\end{matrix}$ is a R-fib [ ] & $\begin{matrix}\Delta^0 \\ \downarrow x \\ X\end{matrix}$ is R-anodyne [4.3.2] ∎

Cor 4.3.8: Let $X \in sSet$, $x \in X_0$. Then id_x is final in $X_{/x}$.

Thm [4.3.9]: Let $X \in qCat$, $x \in X_0$.

The object $\left(\begin{smallmatrix}\Delta^0 \\ \downarrow x \\ X\end{smallmatrix}\right) \in sSet_{/X, \text{contra}}$ has fibrant replacement given by $\left(\begin{smallmatrix}X_{/x} \\ \downarrow \\ X\end{smallmatrix}\right)$.

In particular, $\forall\, y \in X_0$, there's an equiv. of ∞-gpds

$$\mathrm{Map}_{/X}(X_{/x}, X_{/y}) \xrightarrow{\sim} \mathrm{map}_X(x,y).$$

$$\begin{tikzcd} \mathrm{Map}_{/X}(X_{/x}, X_{/y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}(X_{/x}, X_{/y}) \arrow[d, "e_{/y}"] \\ \Delta^0 \arrow[r, "e_{/x}"'] & \mathrm{Hom}(X_{/x}, X) \end{tikzcd}$$

[4.1.12]

$$\begin{tikzcd} \mathrm{Map}_X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}(\Delta^1, X) \arrow[d] \\ \Delta^0 \arrow[r] & X \times X \end{tikzcd}$$

[3.7.1]

This is one form of Yoneda's lemma.

cf. $\mathrm{Hom}_C(A,B) \cong \mathrm{Nat}(h^A, h^B)$.

Pf: By previous cor, $id_x \in (X_{/x})_0$ is a final object.

$$\begin{tikzcd} \Delta^0 \arrow[r, "id_x"] \arrow[dr, "x"'] & X_{/x} \arrow[d] \\ & X \end{tikzcd}$$

is final iff this describes a weak eq. (by def.)

$$\begin{pmatrix} \Delta^0 \\ x \downarrow \\ X \end{pmatrix} \longrightarrow \begin{pmatrix} X_{/x} \\ \downarrow \\ X \end{pmatrix} \quad \text{in } sSet_{/X,\,contra}.$$

4.1.14: For a w.eq. $\Delta^0 \xrightarrow{id_x} X_{/x}$ in $sSet_{/X,\,contra}$.

For any R-fibrant $\begin{matrix} X_{/y} \\ \downarrow \\ X \end{matrix}$, the induced map

$$Map_{/X}(\Delta^0, X_{/y}) \xrightarrow{\sim} Map_{/X}(X_{/x}, X_{/y}) \quad \text{is an eq. of } \infty\text{-gpds.}$$

$$\begin{tikzcd} Map_{/X}(\Delta^0, X_{/y}) \arrow[r] \arrow[d] & Hom(\Delta^0, X_{/y}) \simeq X_{/y} \arrow[d, "p_y"] \\ \Delta^0 \arrow[r, "x"] & Hom(\Delta^0, X) \simeq X \end{tikzcd}$$

$$\Rightarrow Map_{/X}(\Delta^0, X_{/y}) \simeq X_{/y} \times_X \Delta^0 \simeq \underbrace{Map_X(x,y)}_{4.2.10}$$

Prop 4.3.10 $\quad X \in qCat$, $x \in X_0$. $\quad a : id_X \Rightarrow const_x$ a nat'l transf.

i.e. a mor in $Fun(X, X)$

s.t. the induced map

$a_x : x \to x$ is the identity in $ho(X)$

$\Rightarrow x$ is a final object.

skip?

Pf (skipped)

Prop 4.3.11 (Joyal). Let $X \in qCat$.

$x \in X_0$.

$\pi : X_{/x} \to X$ the canonical map

TFAE: (1) x is a final object in X.

(2) $\forall y \in X$, the ∞-gpd $Map_X(y, x)$ is contractible

(3) π is a trivial fibration.

(4) " " equiv. of ∞-cats

(5) π has a section sending $x \mapsto id_x$.

(6) Any map $\partial\Delta^n \xrightarrow{u} X$ w/ $u(n) = x$ fills to a simplex $(n > 0)$

$$\begin{tikzcd} \Delta^0 \arrow[d, "n"'] \arrow[dr, "x"] & \\ \partial\Delta^n \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "u"] \arrow[d] & X \\ \Delta^n \arrow[ur, "\exists"'] & \end{tikzcd}$$

Pf: (1)$\Leftarrow$(5): was 4.3.7.

$$\begin{tikzcd} & (2) \arrow[d, leftrightarrow] & \\ (1) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, "\sim"'] & (3) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & (6) \\ (5) & (4) & \end{tikzcd}$$

(3)$\Longleftrightarrow$(4): since trivial fibs btwn ∞-cats are eq's of ∞-cats. ($sSet_{Joyal}$)

(2)$\Longleftrightarrow$(3): π is a fibration btwn fib. objs in $sSet_{/X, contra}$.

$\Rightarrow$ π is a triv. fib. $\Leftrightarrow$ it's a w.eq. in $sSet_{/X, contra}$.

$\Leftrightarrow$ it's a fiberwise equiv.

4.1.16 $\Leftrightarrow$ $\forall\, y \in X$, fibers are equivalent.

$$\begin{tikzcd} \underset{=}{Map_X(y,x)} \\ (\text{fiber}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{/x} \arrow[d] \\ \Delta^0 \arrow[r, "y"'] & X \end{tikzcd}$$

$\Rightarrow$ $Map_X(y,x) \to \Delta^0$ is a trivial fib. $\Rightarrow$ contractible.

(3)$\Longleftrightarrow$(6): WTS:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X_{/x} \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, "\exists"] & X \end{tikzcd} \quad \text{iff} \quad \begin{tikzcd} \Delta^0 \arrow[dr, bend left, "x"] \arrow[d, hook, "\iota_n"'] & \\ \partial\Delta^n \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, "\exists"'] & \end{tikzcd}$$

$$\begin{tikzcd} \partial\Delta^{n-1} \arrow[r] \arrow[d] & X_{/\Delta^0} = X_{/x} \arrow[d] \\ \Delta^{n-1} \arrow[r] \arrow[ur] & X_{/\emptyset} \times_{\Delta^0_{/\emptyset}} \Delta^0_{/x} \cong X \end{tikzcd} \Longleftrightarrow \begin{tikzcd} (\partial\Delta^{n-1} \star \Delta^0) \cup (\Delta^{n-1} \star \Delta^0) \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^{n-1} \star \Delta^0 \arrow[r] \arrow[ur] & \Delta^0 \end{tikzcd}$$

by Lem. 3.4.20.

$\Leftarrow$:

(1)$\Leftarrow$(3): (3) $\Leftrightarrow$ π a triv. fib $\Rightarrow$ π is final in $sSet_{/X, contra}$.

$$\Rightarrow \quad \begin{tikzcd} \Delta^0 \arrow[r, "x"] \arrow[dr, "id_x"'] & X \\ & X_{/x} \arrow[u, "\pi"'] \end{tikzcd} \quad \text{is final}$$

$\Rightarrow$: If x is a final object $\Rightarrow$ (4) π is an equiv. cats.

$\Leftrightarrow$ (3).

∎

Cor. x a final object in $\underset{\substack{\uparrow \\ q\mathrm{Cat}}}{X}$ $\Rightarrow$ x is a final object in hX.

Cor: The final objects in X form an ∞-gpd which is either
- $\emptyset$
- contractible.

Cor: $x \in X$ final $\Rightarrow$ $\forall$ $A \in \mathrm{sSet}$, the constant functor

$$\mathrm{const}_x : A \longrightarrow X$$

is a final object in $\underline{\mathrm{Hom}}(A, X)$.

skip.

Thm 4.3.16: Let $x \in X$ be an object in an ∞-cat.

TFAE: (1) x is final
(2) $\forall$ $A \in \mathrm{sSet}$, $\mathrm{const}_x : A \to X$ is final in $h(X^A)$.

Rk: Consider a 2-cat. Cat_∞^2 — ob = ∞-cats.
$\forall$ $X, Y \in q\mathrm{Cat}$,
$\mathrm{Hom}_{\mathrm{Cat}_\infty^2}(X, Y) := h(Y^X)$ morphism categories.

The preceding thm says that final objects can be detected in this 2-cat.

This is a special case of the idea that adjunctions btwn ∞-cats are 2-cat'l.
(Riehl-Verity).